The Universal Connection and Metrics on Moduli Spaces

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Abstract

We introduce a class of metrics on gauge theoretic moduli spaces. These metrics are made out of the universal matrix that appears in the universal connection construction of M. S. Narasimhan and S. Ramanan. As an example we construct metrics on the $c_2 = 1$ $SU(2)$ moduli space of instantons on $\mathbb{R}^4$ for various universal matrices.

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1 Introduction

The aim of this paper is to give, more or less, natural metrics on gauge theoretic moduli spaces.

The question of alternate metrics to the standard L^2 metric arose recently in the context of the AdS/CFT correspondence. The moduli space in question is the $c_2 = 1$, $SU(2)$ moduli space of instantons on $\mathbb{R}^4$. The L^2 metric is not 'the right one' in that context, essentially because it does not preserve the conformal invariance inherent in the definition of the moduli space. A rather remarkable alternative is the information metric which is built out of $\operatorname{Tr} F_A * F_A$ and its derivatives with respect to the moduli [3]. The information metric is designed to preserve the conformal invariance of the theory at hand and yields, for the round metric on S^4, the standard Einstein metric on AdS_5. It is remarkable in that if one perturbs the metric on S^4 then to first order in that perturbation the information metric remains Einstein [2]. There are also a host of other small miracles associated with this metric.

On other instanton moduli spaces the information metric fails to be a metric as it becomes highly degenerate. The reason for this is that it is overly gauge invariant, meaning that it is invariant under gauge transformations that depend on the moduli. For example a convenient parameterization of the $c_2 = 1$, $SU(n)$ instanton moduli space on $\mathbb{R}^4$ is the one where the $SU(2)$ instanton is embedded in $SU(n)$ and then one acts with rigid $SU(n)$ gauge transformations to obtain the general instanton. But all the rigid gauge transformations leave $\operatorname{Tr} F_A * F_A$ invariant and so drop right out

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of the information metric leaving us with the $SU(2)$ parameters only. Rather more dramatically one sees that the information metric is in fact zero on moduli spaces of flat or parabolic bundles. So, while the L^2 metric does not keep some of the properties we would like, it does have the advantage of being a metric!

The problem then, as set out at the start of this introduction is to find other natural metrics. The L^2 metric is made out of the gauge connection directly while the information metric is constructed from the curvature 2-form. These are natural objects in the theory and that is why these metrics are also natural. These, however, do not exhaust the natural objects that are available to us.

In 1961 M.S. Narasimhan and Ramanan [4] introduced the concept of a universal connection. This connection plays the same role as that of the universal bundle construction for bundles. Specifically any connection on the Principal bundle of interest can be obtained by pull-back from the universal connection (the bundle itself is obtained by pull-back of the universal bundle). The important conclusion there is that any $U(n)$ connection[1] can be expressed as

$$A = U^\dagger dU \tag{1.1}$$

where U is an $m \times n$ rectangular matrix satisfying

$$U^\dagger . U = \mathbb{I}_n. \tag{1.2}$$

Since any such connection is made up of $n^2 d$ real 'functions' on a d-dimensional manifold it is apparent that there is a raw lower bound on m namely that $2m \geq n(d+1)$. This bound is very difficult to meet. Indeed the bound one gets depends precisely on how the matrices U are constructed.

In any case, the observation of M.S. Narasimhan and Ramanan was turned into a powerful tool for self-dual connections on 4-manifolds in the ADHM construction. That construction amounts to a method for obtaining a from a universal connection the required matrix U.

Infact the universal connection appears in the construction of many moduli spaces. We take the attitude that the U matrices are also natural objects in the gauge theory and so should be used in the construction of metrics. Our construction gives us metrics on moduli spaces where the parameterization of the moduli space is contained in the connection. See our main assumption in the next section. When there are 'matter' or 'Higgs' fields present one should use that data as well in the construction of metrics but this depends on the details of the equations one is trying to solve so we do not enter into this, except loosely in the Conclusions. As an example one can check that for instantons on flat $\mathbb{R}^4$ with U given by the ADHM construction one of the proposed metrics is the non-degenerate AdS_5 metric.

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[1]There is no restriction on the structure group, we have fixed on $U(n)$ here for ease of presentation.

The paper is organized as follows. We delay describing the universal connection construction, and so how one may arrive at the matrices U, till Section 3. Consequently, we require the readers indulgence until then and ask that they take on faith results that will be proven in that section. In the next section we introduce possible metrics on the space of connections and ultimately on the moduli space of interest. In Section 3 the universal connection construction is finally given. An improvement in the Abelian case is also presented. Metrics on the $c_2 = 1$ $SU(2)$ instanton moduli space are considered in detail in Section 4 by way of example of the general construction. Finally, in the Conclusions, we end with many open questions.

2 Universal Metrics

The title of this section is perhaps misleading. We mean that these are metrics made from the matrices U. Suppose that we have a parametrized family of connections, with parameters t^i such that

$$A(t) = U^\dagger(t) dU(t). \tag{2.1}$$

Denote the derivative $\partial/\partial t^i$ by ∂_i. We denote by $\mathcal{G}$ the space of maps from M to G. We also denote by $\mathcal{G}_t$ the space of maps from $T \times M$ to G, where T is the space of moduli. So infact for fixed $t^i \in T$, $g(t, x) \in \mathcal{G}$.

Assumptions: We demand that the parameters are 'honest' parameters, that is, we demand that the parameterization is complete. Furthermore, we will presume that we are working on a smooth part of the moduli space. The two assumptions imply that there are no non-zero vectors $v^i(t)$ such that $v^i(t)\partial_i A(t) = 0$ at any point t^i of the moduli space under consideration. We will refer to these assumptions as the main assumption.

Introduce the projector,

$$P = U.U^\dagger \tag{2.2}$$

which is an $m \times m$ hermitian matrix, satisfying

$$P^2 = P.$$

The projector, P, is not only gauge invariant since $U \to U.g^\dagger$ for $g \in \mathcal{G}$ but also invariant under $\mathcal{G}_t$. We have the following simple,

Lemma 2.1 If $\partial_i P = 0$ then $\partial_i A = d_A A_i$; where $A_i = U^\dagger \partial_i U$ and the covariant derivative is $d_A = d + [A, .$

Proof: This is by direct computation. $\partial_i P = 0$ implies that

$$\partial_i U^\dagger = -U^\dagger \partial_i U U^\dagger, \quad \partial_i U = -U \partial_i U^\dagger U.$$

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For the variation of the connection we have, thanks to these two equations,

$$
\begin{aligned}
\partial_i A &= \partial_i U^\dagger . dU + U^\dagger d \partial_i U \\
&= -U^\dagger \partial_i U\, A + A\, U^\dagger \partial_i U + d\left(U^\dagger \partial_i U\right) \\
&= d_A A_i
\end{aligned}
$$

$\square$

One can consider the connection form to be a form in a higher dimension, that is on $T \times M$, with the components in the T direction being $A_i = U^\dagger \partial_i U$. With this understood Lemma 2.1 says that $v^i \partial_i P = 0$ only if $v^i F_{i\,\mu} = 0$ ($F_{i\,\mu}$ are the mixed components of the curvature 2-form on $T \times M$).

We now introduce some universal metrics. Set

$$
g_{ij}^0 = \int_M \mathrm{Tr}\,(\partial_i P * \partial_j P). \tag{2.3}
$$

$$
g_{ij}^1 = -\int_M \mathrm{Tr}\, A_i * A_j, \tag{2.4}
$$

where $*$ is the Hodge star operator. g^0 is invariant under $\mathcal{G}_t$ while g^1 is only invariant under $\mathcal{G}$. The metrics that one gets naturally descend to $\mathcal{A}/\mathcal{G}$ and finally to the moduli space.

Theorem 2.2 Let M be a compact closed manifold and U a universal matrix for some family of connections then there is a linear combination of the components of the quadratic forms g^0 (2.3) and g^1(2.4) which is a metric on the moduli space.

Proof: There are two cases to consider:

First suppose that there is no vector v^i such that $v^i \partial_i P = 0$, then g^0 is a metric. This follows from the fact that $*$ and Tr (on Hermitian matrices) are positive definite so that for g^0 to be degenerate, that is for $g_{ij} v^i v^j = 0$ for some v^i, we must have $v^i \partial_i P = 0$.

Secondly suppose that there is a vector v^i such that $v^i \partial_i P = 0$ then g^0 is degenerate in this direction. However, by Lemma 2.1 we have that $v^i \partial_i A = d_A\, v^i A_i$ and $v^i A_i$ cannot be zero as that would contradict our main assumption. Positive definiteness of $*$ and negative definiteness of Tr (on anti-Hermitian matrices) guarantee that $g_{ij}^1 v^i v^j \neq 0$.

Consequently we can always organize for some linear combination of the components of g^0 and g^1 to yield a non-degenerate symmetric quadratic form on T.

$\square$

We have a kind of converse to the theorem,

Corollary 2.3 (to Theorem 2.2) Given the conditions of the theorem if $A_i = 0$ then g^0 is a metric on the moduli space.

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Proof: If $A_i = 0$ then there is no v^i such that $v^i \partial_i P = 0$ since if there was we would conclude $v^i \partial_i A = 0$ contradicting our main assumption.

$\square$

To see that we can apply Corollary 2.3 of Theorem 2.2 directly we quote a

Lemma 2.4 If U is the NR (M.S. Narasimhan, Ramanan) matrix then

$$A_i = U^\dagger \partial_i U = 0.$$

Proof: This is delayed till Section 3 where all the definitions will also be available.

As the construction in [4] applies to any connection we learn, from Corollary 2.3, that any moduli space can be given the metric g^0 providing the universal matrix used is a NR matrix.

If one wishes to make use of U matrices other than those which are NR matrices then (2.3) can fail to be, but need not fail to be, a metric. In the proof of Theorem 2.2 we saw that degeneracy of g^0 only comes from having connections whose dependence on some moduli is through gauge transformations. This is precisely the situation that we described in the Introduction for the data for instantons for higher rank and which plagues the information metric.

In cases of this type one has that some of the moduli, say s^a, are obtained by a gauge transformation on a connection A_0 which depends on moduli r^α, that is $A(s, r) = (U_0(r)h(s, r))^\dagger d(U_0(r)h(s, r)) = h^\dagger(s, r)A_0(r)h(s, r) + h^\dagger(s, r)dh(s, r)$ then $\partial_a A(s, r) = d_A A_a(s, r)$ where $A_0(r) = U_0^\dagger(r)dU_0(r)$ and $A_a(s, r) = h^\dagger(s, r)\partial_a h(s, r)$. Furthermore, we have that P depends on all the coordinates r^α but $\partial_a P = 0$. Now $g^0_{ab} = g^0_{a\alpha} = 0$ however,

$$g^1_{ab} = -\int_M \mathrm{Tr}\left(h^\dagger(s, r)\partial_a h(s, r) * h^\dagger(s, r)\partial_b h(s, r)\right).$$

Thus, in this situation, one can consider as a metric a linear combination of g^0 and g^1.

Proposition 2.5 Let M be a compact, closed manifold and U a universal matrix for a family of connections with $U = U_0(r)\, h(s, r)$ where g^0 for U_0 is a metric on the part of the moduli space parameterized by r^α and $h \in \mathcal{G}_s$ as above, then a linear combination of g^0 and g^1_{ab} is a metric on the moduli space.

$\square$

Remark 2.6 Notice that we are not saying that $A_\alpha = g^\dagger(s, r)\partial g / \partial r^\alpha$ is zero. Such a condition is not required for g^0 to be a metric.

2.1 More Metrics from the Universal Connection

One of the problems we are faced with is that for non-compact manifolds the integrals that go into defining g^0 and g^1 may well not converge. To improve the situation we

add a 'damping' factor. This generalization gives us many possible metrics even in the compact case.

Let

$$\Phi(U) = *^{-1}\operatorname{Tr}(dP * dP) \tag{2.5}$$

So far we had only used a volume form on M, however $\Phi(U)$ requires a metric. Note that $\Phi(U)$ is invariant under $\mathcal{G}_t$.

As an aside note that the mass dimension of $\Phi(U)$ is 2 which means that it is like a mass term for the gauge field A_μ. Infact it is gauge invariant albeit highly non-local and non-polynomial (in the gauge field). One has

$$\int_{\mathbb{R}^4} d^4x \; \Phi(U) = \int_{\mathbb{R}^4} d^4x \; \operatorname{Tr}\left(-A_\mu\, A^\mu + \partial_\mu U^\dagger\, \partial^\mu U\right).$$

which is much more suggestive of a mass for the gauge field.

Set

$$g_{ij}^{0,\,\alpha} = \int_M \Phi(U)^\alpha \; \operatorname{Tr}(\partial_i P * \partial_j P), \tag{2.6}$$

$$g_{ij}^{1,\,\beta} = -\int_M \Phi(U)^\beta \operatorname{Tr}\left(U^\dagger\partial_i U * U^\dagger\partial_j U\right). \tag{2.7}$$

Proposition 2.7 $g^{0,\,\alpha}$ and $g^{1,\,\beta}$ clearly have the following properties.

1. They are gauge invariant (that is under $\mathcal{G}$).

2. For $\alpha = \beta = d/2$ where $\dim M = d$, the metric on M enters only through its conformal class.

3. For M non-compact with α and β suitably large the integrals in (2.6, 2.7) formally converge provided that $\Phi(U)$ has some suitable integrability properties.

$$\square$$

As far as the third point of the proposition is concerned it may be natural to suppose that $\Phi(U)$ be integrable (as indicated by the analogy with a mass term). However, weaker integrability conditions may also suffice in certain cases.

3   The Universal Connection

M. S. Narasimhan and Ramanan [4] prove that, at least locally, any connection, A on a bundle $\mathcal{P}$ can be expressed as

$$A = iU^\dagger dU. \tag{3.1}$$

The set up is as follows [4]. One begins with the Stiefel manifold, $V(m, n)$ of all unitary n-frames through the origin of $\mathbb{C}^m$, thought of as a $U(n)$ principal bundle over the Grassman manifold $G(m, n) \equiv U(m)/(U(m) \times U(m-n))$. There is a connection on this bundle which has the following description. Denote the components of the n-planes by $v_i = \sum_{a=1}^{m} e_a S_{ai}$ where the (e_a) are a canonical basis for $\mathbb{C}^m$ and $i = 1, \ldots, n$. By orthogonality one requires that $S^\dagger . S = \mathbb{1}_n$ (the unit $n \times n$ matrix). Denote the matrix valued function which associates to each n-plane its matrix S_{ai} by the same letter. Then

$$S^\dagger \, dS = \omega \tag{3.2}$$

is a canonical connection on the bundle. This is the universal connection. The main theorem of M.S. Narasimhan and S. Ramanan is that any connection on a principle $U(n)$-bundle, $\mathcal{P}$ over a d-dimensional manifold X, is obtained by pullback of ω from a differentiable bundle homomorphism from $\mathcal{P}$ to $V(m, n)$ for some sufficiently large m.

The theorem tells us that any connection may be expressed in the form (3.1) for an $m \times n$ matrix U providing that m is sufficiently large. They provide a lower bound on m, $m \geq (d+1)(2d+1)n^3$ will do, but it is not a very efficient one. We will see below that for the local problem for $U(1)$ bundles one can infact do much better than requiring $m = (d+1)(2d+1)$.

3.1 The Narasimhan-Ramanan Matrix

In their paper [4] Narasimhan and Ramanan not only prove the existence of the universal connection but also give a construction of the matrices U. The way they do this is to pass from a local construction of U to a more global one. They certainly do not give the most minimal form of U but, nevertheless, their procedure is the only one we know of that will produce the required matrix U for any connection.

Lemma 3.1 [4] Let V be an open subset of $\mathbb{R}^d$ and W a relatively compact open subset whose closure is contained in V. For every differential form α of degree 1 on V with values in $\mathfrak{u}(n)$ (the space of skew-Hermitian matrices), there exist differentiable functions $\phi_1, \cdots \phi_{m'}$ in W with values in the space $\mathcal{M}_n(\mathbb{C})$ of $(n \times n)$ complex matrices such that

1. $\sum_{j=1}^{m'} \phi_j^* \phi_j = \mathbb{1}_n$, and

2. $A = \sum_{j=1}^{m'} \phi_j^* d\phi_j$.

where $m' = (2d+1)n^2$.

The Narasimhan-Ramanan matrices are of a very particular form. Write

$$A = i \sum_{\mu=1}^{d} \sum_{r=1}^{n^2} \lambda_{r,\mu} f_r dx_\mu \tag{3.3}$$

where $(f_1 \cdots\cdots f_{n^2})$ a set of positive define matrices which form a base for the complex Hermitian matrices over the reals, such that $||f_r|| = 1$ for every r, here $||f||$ being the norm as a linear transformation. According to the proof the lemma, there exists $p_{r,\mu}$. $q_{r,\mu}$ and h_r strictly positive differentiable functions such that

$$\lambda_{r,\mu} = p_{r,\mu}^2 - q_{r,\mu}^2,$$

$$h_r^2(x) = \frac{1}{n^2}\mathbb{1}_n - \left[\sum_{\mu=1}^{d}(p_{r,\mu}^2 + q_{r,\mu}^2)\right]f_r \qquad (3.4)$$

The matrix U that Narasimhan and Ramanan propose is,

$$U(A) = \begin{bmatrix} \Phi_1 \\ \Phi_2 \\ \Phi_3 \end{bmatrix}, \qquad (3.5)$$

where Φ_1 has $d(n \times n)$ components defined by the functions $p_{r,\mu}e^{ix_\mu} \cdot g_r$, Φ_2 has $d(n \times n)$ components defined by the functions $q_{r,\mu}e^{-ix_\mu} \cdot g_r$, Φ_3 has $(n \times n)$ functions defined by h_r and g_r is the positive square root of f_r. Think of all of these as vectors with entries $(n \times n)$ matrices.

There are two, somewhat surprising, results that can be deduced from the lemma and its proof.

Proposition 3.2 In general the Narasimhan-Ramanan matrix for the gauge transform of a connection is not the gauge transformation of the Narasimhan-Ramanan matrix of the original connection, i.e. it is not equivariant,

$$U(A^g) \neq U(A).g \qquad (3.6)$$

Lemma 2.4 Let $A(t)$ be a family of connections parameterized by t^i and let $U(t)$ be the corresponding family of Narasimhan-Ramanan universal matrices. Then,

$$U^\dagger(A)\partial_i U(A) = 0. \qquad (3.7)$$

where $\partial_i = \partial/\partial t^i$.

The first proposition is evident from the construction of the matrix $U(A)$ (one should refer to [4] for the details of that construction).

Proof of Lemma 2.4:

From the definitions we have,

$$\Phi_1^\dagger \partial_i \Phi_1 = \frac{1}{2}\sum_{r,\mu}\partial_i p_{r,\mu}^2 f_r, \quad \Phi_2^\dagger \partial_i \Phi_2 = \frac{1}{2}\sum_{r,\mu}\partial_i q_{r,\mu}^2 f_r, \quad \Phi_3^\dagger \partial_i \Phi_3 = \frac{1}{2}\sum_r \partial_i h_r^2,$$

so that

$$\sum_{a=1}^{3} \Phi_a^\dagger \partial_i \Phi_a = \partial_i \frac{1}{2}\left(\sum_{r,\mu}(p_{r,\mu}^2 + q_{r,\mu}^2)f_r + \sum_r h_r^2\right) = \partial_i \frac{1}{4n^2}\mathbb{1}_{n\times n} = 0.$$

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$\square$

From Proposition 3.2 we learn that there are universal parameterizations which are not gauge covariant. In particular, it is difficult to see how to construct invariants from the Narasimhan-Ramanan matrix apart from those made out of the curvature 2-form. This suggests that, in this case, one should already work on a slice of the space of connections with this parameterization.

Somewhat more mysterious is Lemma 2.4. It implies a special case of Proposition 3.2. Suppose that U is some parameterization matrix, not necessarily the NR matrix, which satisfies (3.7) and is covariant $U(A^g) = U(A).g$ for all $g \in \mathcal{G}_t$ then we get into a contradiction. Inserting the covariance condition into (3.7) we find that

$$g^\dagger \partial_i \, g = 0,$$

which implies that g cannot be an arbitrary gauge transformation but rather only one that does not depend on the parameters t^i.

3.2 Abelian Universal Connections

The NR matrices are very messy to deal with, as can be seen from the details of their construction. However, in the Abelian case one can simplify the discussion and construction somewhat.

Lemma 3.3 There exists fixed real-valued functions r_i, for $i = 1, \ldots, d-1$, on $\mathbb{R}^d$ such that any $U(1)$ connection A, which is pure gauge at infinity, can be expressed as

$$A = -\sum_{i=1}^{d-1} \theta_i \, dr_i^2 + d\bar\theta. \tag{3.8}$$

for some functions $\bar\theta$ and θ_i, with $i = 1, \ldots, d-1$.

Proof: Let $\partial_j r_i^2 = \delta_{ij} \partial_i r_i^2$ for $i = 1, \ldots, d-1$ and such this derivative does not vanish anywhere except at infinity, i.e. $\partial_i r_i^2 \to 0$ as $|x| \to \infty$. This means that we can invert $\partial_i r_i^2$ everywhere. Let A match $d\bar\theta$ at infinity. Set,

$$\theta_i = -\frac{1}{\partial_i r_i^2} \left(A_i - \partial_i \bar\theta_d \right)$$

and in any case we have that $A_d = \partial_d \bar\theta$, from which we can solve for $\bar\theta$. Plugging back in establishes the lemma.

A set of functions that have the required property are $r_i^2 = c_i (\exp(x_i) + 1)^{-1}$ where c_i are constants.

$\square$

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Corollary 3.4 A Universal matrix for any $U(1)$ connection A on $\mathbb{R}^d$, which is pure gauge at infinity, is given by

$$U = \begin{bmatrix} r_1 e^{-i(\theta_1+\theta_d)} \\ r_2 e^{-i(\theta_2+\theta_d)} \\ \vdots \\ r_{d-1} e^{-i(\theta_{d-1}+\theta_d)} \\ r_d e^{-i\theta_d} \end{bmatrix},$$

where $r_d^2 = 1 - \sum_{i=1}^{d-1} r_i^2$. and $\theta_d = \bar{\theta} - \sum_{i=1}^{d-1} r_i^2 \theta_i$.

Proof: We have that

$$A = iU^\dagger dU = \sum_{i=1}^{d-1} r_i^2 d(\theta_i + \theta_d) + d\theta_d = -\sum_{i=1}^{d-1} dr_i^2\, \theta_i + d\bar{\theta}$$

and by the lemma this is the general form for any such connection.

$\square$

The NR matrices, in the $U(1)$ case, are $2d + 1 \times 1$ matrices. The connection matrix above is a $d \times 1$ matrix and since a connection form is essentially given by d functions we may consider this to be an optimal parameterization.

4 Instanton Moduli Space

As already mentioned a convenient parameterization of the $c_2 = 1$, $SU(n)$ instanton moduli space on $\mathbb{R}^4$ is the one where the $SU(2)$ instanton is embedded in $SU(n)$ and then one acts with rigid $SU(n)$ gauge transformations to obtain the general instanton. From previous sections we know that the metric coming from these gauge transformations will be caught by g^1 and the $SU(2)$ part will come from g^0. In this section we concentrate on the $SU(2)$ part.

The moduli space $\mathcal{M}^1_{SU(2)}$ is known to be five-dimensional, with the topology of the open 5-ball, thus parametrized by five moduli, namely four coordinates a^μ (the centre of the instanton) and one scale ρ. The parameter ρ measures the size of the instanton, and zero size corresponds to delta function or so called singular instantons. $\rho = 0$ corresponds to the boundary of the 5-ball, S^4, where the moduli a^μ are the coordinates on the S^4.

The following sections are relatively brief as all the details are essentially computational and many steps are skipped.

4.1 The ADHM Universal Matrix and its Metric

The ADHM construction gives the universal connection form for the instanton that we are interested in [1]. These authors find a 4×2 representation for the rectangular

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matrices required to parameterize an instanton, namely the explicit expression of U for the self-dual gauge potential is

$$U(x) = \frac{1}{\sqrt{(x-a)^2 + \rho^2}} \begin{bmatrix} (x-a)^\mu \bar{\sigma}_\mu \\ \\ -\rho \mathbb{1}_{2\times 2} \end{bmatrix} \tag{4.1}$$

We use the notation

$$\sigma_\mu = (\mathbb{1}_{2\times 2}, i\tau_a), \quad \bar{\sigma}_\mu = (\mathbb{1}_{2\times 2}, -i\tau_a), \tag{4.2}$$

with τ_a the usual Pauli matrices.

A straightforward calculation leads to

$$\text{Tr}\,(\partial_{a^\mu} P\, \partial_{a^\nu} P) = \frac{4\delta_{\mu\nu}\,\rho^2}{\left((x-a)^2 + \rho^2\right)^2}$$

$$\text{Tr}\,(\partial_\rho P\, \partial_{a^\nu} P) = \frac{4\rho\,(x-a)_\nu}{\left((x-a)^2 + \rho^2\right)^2}$$

$$\text{Tr}\,(\partial_\rho P\, \partial_\rho P) = \frac{4(x-a)^2}{\left((x-a)^2 + \rho^2\right)^2}. \tag{4.3}$$

We do not have to work to calculate $\Phi(U)$. By translational invariance one can replace derivatives with respect to x^μ with derivatives with respect to $-a^\mu$ so that we find

$$\text{Tr}\,\Phi(U) = \frac{16\rho^2}{\left((x-a)^2 + \rho^2\right)^2} \tag{4.4}$$

Our next task is to determine the metric $g^{0,\alpha}$. For $\alpha > 1/2$ the integrals converge and we consider α in this range. By rotational invariance the integral defining $g^{0,\alpha}_{\rho\,a^\mu}$ is zero. Likewise, by translational invariance, the other integrals do not depend on a^μ. The metric on the moduli space is, therefore, of the form

$$ds^2 = \rho^{1-\alpha}\,A(\alpha)\left(d\vec{a}^2 + B(\alpha)d\rho^2\right) \tag{4.5}$$

where the dependence on ρ is determined by dimensional arguments. The coefficients $A(\alpha)$ and $B(\alpha)$ are determined on doing the integrals and they are both non-zero, thus this is a metric. When $\alpha = 1$ this is proportional to the L^2 metric. When $\alpha = 4/2$, ds^2 is proportional to the AdS$_5$ metric and this had to be so as a consequence of Proposition 2.7. Thus with $\alpha = 2$ the universal metric exhibits the nice feature of the information metric namely that it is Einstein.

For completeness we list the values of the coefficients

$$A(\alpha) = (4)^{2\alpha+1}\,\pi^2\,[2\alpha - 1]\,\frac{\Gamma(2\alpha - 1)}{\Gamma(2(\alpha+1))}$$

$$B(\alpha) = \frac{2}{2\alpha - 1}.$$

4.2 The NR Universal Matrix and its Metric

The construction of the NR matrix is somewhat involved, so here we will outline some of the ingredients and leave the details for [5].

For the instanton moduli space $\mathcal{M}^1_{SU(2)}$, the explicit expression for the connection of interest, A_μ, is given by

$$A_\mu = \eta^a_{\mu\nu} \frac{(x-a)_\nu}{(x-a)^2 + \rho^2} \tau_a, \tag{4.6}$$

where $\eta^a_{\mu\nu}$ are the 't Hooft eta-symbols, a basis for self-dual two-form on $\mathbb{R}^4$, and τ_a are the Pauli matrices. The basis of positive definite Hermitian matrices f_r are chosen to be $f_i = (\tau_i + 3\mathbb{I}_2)/4$ for $i = 1,\ 2,\ 3$ and $f_4 = \mathbb{I}_2$. The $\lambda_{r,\mu}$ are given by the expressions

$$\lambda_{i,\mu} = 4\eta^i_{\mu\nu} \frac{(x-a)_\nu}{(x-a)^2 + \rho^2}, \quad i = 1,\ 2,\ 3 \tag{4.7}$$

$$\lambda_{4,\mu} = -\frac{3}{4} \sum_{i=1}^{3} \lambda_{i,\mu}. \tag{4.8}$$

One can see directly from the construction of $p_{r\mu}$ and $q_{r\mu}$, as spelled out in the proof of the Lemma on page 565 of [4], that these functions depend on the position of the instanton, a^μ, only through the combination $(x-a)^\mu$. This is due to the fact that the $a_{r,\mu}$ (that appear on page 566), for the single $SU(2)$ instanton, are functions of the scale ρ and not of a^μ. Consequently, once more by dimensional arguments, the $g^{0\alpha}$ metric will take the form given in (4.5) and only the coefficients $A(\alpha)$ and $B(\alpha)$ need to be determined.

This agreement of g^0, up to change of parameters, of the ADHM and NR universal matrices is not a 'generic' situation. The proof of the Lemma just cited holds if the $a_{r,\mu}$ are replaced by $a_{r,\mu} + k_{r,\mu}$ where the $k_{r,\mu}$ are positive functions. For example one could consider $k_{r,\mu} = c_{r,\mu} \exp(-d_{r,\mu} a^2/\rho^2)$ with $c_{r,\mu}$ and $d_{r,\mu}$ positive constants satisfying suitable conditions so that h_r is well defined. In this case the universal matrix would lead to a metric with a highly non-trivial dependence on a^2.

5 Conclusions

There are many more metrics that one can form from the universal matrices U. For example, both the L^2 and information metrics can be written in terms U. One can also construct other 'damping' factors. If we set

$$\phi = *^{-1} dP * dP$$

then powers of terms of the form

$$\mathrm{Tr}\,\phi^{\alpha_1} \ldots \mathrm{Tr}\,\phi^{\alpha_k}$$

with α_i positive integers, give us more positive definite damping factors. Proposition 2.7 holds if one replaces Φ with the damping factors we have just introduced.

One question that has not been addressed is: how are metrics that come from different parameterizations, meaning different universal matrices, related? In some cases are they related simply by a diffeomorphism? We do not have an answer to this series of questions. However, there is a set of cases in this direction where we have a simple

Proposition 5.1 Let U be a $m \times n$ universal matrix and M a $k \times n$ universal matrix with $k \geq m$ so that
$$A_\mu = iU^\dagger \partial_\mu U = iM^\dagger \partial_\mu M.$$

If
$$M = \frac{1}{\sqrt{N}} \begin{bmatrix} U \\ U \\ \vdots \\ U \end{bmatrix},$$

with $k = Nm$, then the universal metrics of U and M agree.

$\square$

We have also not investigated the geometry of the metrics that have been introduced. This appears to require a more detailed knowledge of the moduli space that one is dealing with as, for example, of the instanton moduli space of the previous section.

Many gauge theory moduli spaces also involve 'matter' fields and consequently any metric on the moduli space would probably need to involve those objects as well. Suppose Ψ is such a field, that is a section of some associated bundle to the Principal bundle (tensored with other bundles). Think of Ψ as being valued in some tensor product representation $V \otimes \ldots \otimes V$ where V is the n-dimensional representation of $SU(n)$. In this case the covariant derivative

$$d_A \Psi = U^\dagger \otimes \ldots \otimes U^\dagger \, d \, (U \otimes \ldots \otimes U \, \Psi)$$

and one can form gauge invariant combinations $U \otimes \ldots \otimes U \, \Psi$ and from this construct gauge invariant terms to be added to the universal metric.

Finally, we would also like to know if there exists a construction which produces equivariant universal matrices. We note that the U matrices in the Abelian case in Corollary 3.4 are indeed equivariant.

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